ELSEVIER Journal of pure and applied algebra 108 (1996) 265–278

JOURNAL OF PURE AND APPLIED ALGEBRA

An example of a nontrivial ring topology on the algebraic closure of a finite field

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Communicated by M.-F. Roy; received 12 December 1994, revised 2 January 1995

Abstract

A description is given of a nondiscrete ring topology on the algebraic closure of a finite field. Its completion is constructed, and a family of continuous functions on it is given. Some generalizations to other fields are suggested.

1. Introduction

All fields considered in this paper are assumed to be commutative. In 1968, Kiltinen [7] and Arnautov [1] proved that every infinite field admits a proper field topology. They make use of an inductive procedure which was first introduced by Hinrichs [6]. A field is called algebraic if it is an algebraic extension of a finite field. In [7], it is proved that the only fields which do not admit a proper locally bounded ring topology are the algebraic fields. In this article we introduce a ring topology for the algebraic closure $\Gamma(p)$ of a finite field of characteristic p. This topology is described in a more explicit way than the inductive topologies in [1, 7, 13, p. 84]. We also study the completion of $\Gamma(p)$, which we denote by Λ_p; in this ring we introduce a family of continuous functions with good convergence properties. Next, we introduce a slightly different topology with similar features. Finally, we deal with generalizations of our topology to other fields. For a general background about algebraic fields, the book [3] is recommended.

Both topologies introduced here were first obtained by means of nonstandard analysis (see [11, Section 4]); we follow standard treatment in order to get a more accessible exposition.

SSDI 0022-4049(95)00044-5

2. Construction of the topology

Throughout the paper, let p be a fixed prime number. Let $GF(p^n)$ be the finite field with p^n elements, and let $\Gamma(p)$ be its algebraic closure, which we suppose to contain $GF(p^n)$ for all $n \in \mathbb{N}$. Let us see some properties satisfied by these fields. The field $GF(p^n)$ is a subfield of the field $GF(p^m)$ if and only if $n|m$. The field $\Gamma(p)$ can be written as the union

$$\Gamma(p) = \bigcup_{n=1}^{+\infty} GF(p^{n!}) = \bigcup_{n=1}^{+\infty} GF(p^n).$$

For all $n \geq 2$, we have that

$$GF(p^{n!})/GF(p^{(n-1)!}) \tag{1}$$

is a simple algebraic extension field of degree n. For each $n \geq 2$, we fix an irreducible polynomial $P_n(X) \in GF(p^{(n-1)!})[X]$ of degree n such that we have the field isomorphism

$$GF(p^{n!}) \cong \frac{GF(p^{(n-1)!})[X]}{(P_n(X))}.$$

We fix a root of each polynomial $P_n(X)$, which we will denote by $\alpha_n \in GF(p^{n!})$. This α_n is a primitive element of the extension field (1), and therefore we have

$$GF(p^{n!}) = GF(p^{(n-1)!})[\alpha_n]. \tag{2}$$

Furthermore, we will call $P_1(X) = 1$ and $\alpha_1 = 1$. Throughout this article, $\{P_n(X)\}_{n \in \mathbb{N}}$ and $\{\alpha_n\}_{n \in \mathbb{N}}$ will denote the above family of polynomials and the corresponding family of primitive elements of each extension field (1).

For each $a \in GF(p^{n!})$ there is a unique polynomial

$$Q_a(X) \in GF(p^{(n-1)!})[X] \tag{3}$$

that satisfies $\deg(Q_a(X)) \leq n-1$, and $a = Q_a(\alpha_n)$. For all natural numbers $n \geq 2$ we define

$$G_n = \{a \in GF(p^{n!}) \colon Q_a(X) \text{ satisfies } Q_a(0) = 0\}; \tag{4}$$

namely, G_n is the additive subgroup of $GF(p^{n!})$ consisting of those elements a having associate polynomial $Q_a(X)$ with constant term null. We set $G_1 = GF(p)$. We have the following direct sums of additive subgroups:

$$GF(p^{n!}) = GF(p^{(n-1)!}) \oplus G_n \quad \text{for all } n \geq 2. \tag{5}$$

$$\Gamma(p) = \bigoplus_{n \in \mathbb{N}} G_n. \tag{6}$$

We shall call $H_n = \oplus_{j \geq n} G_j$. Therefore for all $n \geq 2$ we have $\Gamma(p) = GF(p^{(n-1)!}) \oplus H_n$. The inclusion $G_n G_m \subseteq G_m$ holds for all $m > n$. In view of the above, we conclude the following result.

Lemma 1. *Every element $a \in \Gamma(p)$ can be expressed as a sum $a = \sum_{n=1}^{j} a_n$, where $a_n \in G_n$, and such representation is unique.*

Let us define two functions on $\Gamma(p)$.

Definition 1. Let a $\in \Gamma(p)$ be a nonzero element such that $a = \sum_{n=1}^{j} a_n$, where $a_n \in G_n$. We define

$$\text{depth}(a) = \min\{n\colon a_n \neq 0\} = \max\{n\colon a \in H_n\},$$

$$\text{height}(a) = \max\{n\colon a_n \neq 0\} = \min\{n\colon a \in GF(p^{n!})\}.$$

Further we set depth(0) = height(0) = 1.

We shall also define another integer function for each subgroup G_n, which we shall call "weak degree", and we denote by "wd( )".

Definition 2. For each $n \geq 2$, let $a \in G_n \backslash \{0\}$, and let $Q_a(X) \in GF(p^{(n-1)!})[X]$ be the polynomial defined in (3); we shall call $\text{wd}(a) = \deg(Q_a(X))$. We complete the definition with $\text{wd}(a) = 1$ for all $a \in GF(p)$.

It is clear that $0 < \text{wd}(a) \leq n - 1$ for all $a \in G_n$ with $n \geq 2$. We have called it "weak degree" in order to avoid confusion with the degree of an element a belonging to $\Gamma(p)$. When $a \in \Gamma(p)$ the expression *degree*(a) usually means the degree of the field extension $GF(p)[a]/GF(p)$, i.e., the degree of the irreducible polynomial of a over the prime field $GF(p)$. The following two lemmas are immediate consequences of the definitions.

Lemma 2. *For every $a, b \in \Gamma(p) \backslash \{0\}$, the functions "depth" and "height" defined above have the following properties:*

$$\text{depth}(a+b) \geq \min\{\text{depth}(a), \text{depth}(b)\}, \quad \textit{equality holds if } \text{depth}(a) \neq \text{depth}(b),$$

$$\text{height}(a+b) \leq \max\{\text{height}(a), \text{height}(b)\}, \quad \textit{equality holds if } \text{height}(a) \neq \text{height}(b).$$

In addition, if $a = \sum_{n=1}^{j} a_n$ and $b = \sum_{n=1}^{j} b_n$, where $a_n, b_n \in G_n$ such that $\text{wd}(a_n) < n/2$ and $\text{wd}(b_n) < n/2$ for all $n \leq j$, then

$$\text{depth}(ab) = \max\{\text{depth}(a), \text{depth}(b)\},$$

$$\text{height}(ab) = \max\{\text{height}(a), \text{height}(b)\}.$$

Lemma 3. *For all $a, b \in G_n$, the function "weak degree" satisfies that $wd(a+b) \leq \max\{\text{wd}(a), \text{wd}(b)\}$, and equality holds if $wd(a) \neq \text{wd}(b)$. Furthermore, if $wd(a) < n/2$ and $wd(b) < n/2$, then $wd(ab) \leq \text{wd}(a) + \text{wd}(b)$, with equality holding if $ab \neq 0$.*

With the aid of these functions we are prepared to define a neighborhood basis at zero of a ring topology on $\Gamma(p)$. For all $m \in \mathbb{N}$ let

$$V_m = \left\{ \sum_{n \geq m} a_n \in \Gamma(p)\text{: for all } n,\ a_n \in G_n \text{ and } \text{wd}(a_n) \leq \frac{n}{m} \right\}. \tag{7}$$

This family of neighborhoods of zero will be denoted by $\mathscr{B} = \{V_m\}_{m\in\mathbb{N}}$. Every nonzero element $a \in V_m$ has depth(a) greater than or equal to m. Thus for all $m \in \mathbb{N}$ we have the inclusion $V_m \subset H_m$. The following property can be easily checked.

Lemma 4. *Each neighborhood of zero V_m is an additive subgroup of $\Gamma(p)$.*

We recall that for a sequence $\{U_n\}_{n\in\mathbb{N}}$ of subsets of a commutative ring R to be a fundamental system of neighborhoods at zero for a Hausdorff ring topology on R, it suffices that the following properties hold.

$$\text{For all } n,\quad 0 \in U_n,\ U_n = -U_n,\ U_{n+1} = U_n. \tag{8}$$

$$\text{For all } n,\quad U_{n+1} + U_{n+1} \subset U_n. \tag{9}$$

$$\text{For all } n \text{ there exists } k \text{ such that } U_{n+k}U_{n+k} \subset U_n. \tag{10}$$

$$\text{For all } n \text{ and } x \in R \text{ there exists } k \text{ such that } xU_{n+k} \subset U_n. \tag{11}$$

$$\bigcap_{n\in\mathbb{N}} U_n = \{0\}. \tag{12}$$

See, for instance, [7, 9, 12, 14]. We can now show the following result.

Theorem 5. *The family $\mathscr{B} = \{V_m\}_{m\in\mathbb{N}}$ defined in (7) is a basic system of neighborhoods of zero for a Hausdorff ring topology on $\Gamma(p)$, that we will denote by $\mathscr{T}$.*

Proof. We are going to see that this family $\mathscr{B}$ satisfies properties (8)–(12). It is obvious that (8) holds; (9) is a consequence of every V_m being an additive group. We now check (11). Given $V_m \in \mathscr{B}$ and $a \in \Gamma(p)$, with $a \neq 0$; we take $k \in \mathbb{N}$ such that $m + k >$ height(a). We claim that $aV_{m+k} \subset V_m$. Let $b \in V_{m+k}$ with $b \neq 0$; we can express $b = \sum_{n=m+k}^{j} b_n$, where $b_n \in G_n$. Since $n \geq m + k >$ height(a), then $ab_n \in G_n$, and we can write $ab = \sum_{n=m+k}^{j} ab_n$. We have depth$(ab) =$ depth$(b) \geq m + k$, and for all n such that $m + k \leq n \leq j$, the equality $\mathrm{wd}(ab_n) = \mathrm{wd}(b_n)$ holds. Consequently, $ab \in V_{m+k} \subset V_m$, and the claim is proved.

Let us show (10). Given $V_l \in \mathscr{B}$, we take an integer m such that $m \geq 2l$ and $m \geq 3$; we claim that $V_mV_m \subset V_l$. For all $a, b \in V_m$ with $a \neq 0$ and $b \neq 0$, we give their representation $a = \sum_{n=m}^{j} a_n$ and $b = \sum_{n=m}^{k} b_n$, where $a_n,\ b_n \in G_n$, $\mathrm{wd}(a_n) \leq n/m \leq n/3$ and $\mathrm{wd}(b_n) \leq n/m \leq n/3$. Then,

$$ab = \sum_{n=m}^{\max(j,k)} c_n \quad \text{where } c_n = \left(\sum_{i=m}^{n-1} a_i\right)b_n + \left(\sum_{i=m}^{n-1} b_i\right)a_n + a_nb_n \in G_n.$$

For all n such that $m \leq n \leq \max(j, k)$, the following inequality holds:

$$\mathrm{wd}(c_n) \leq \mathrm{wd}(b_n) + \mathrm{wd}(a_n) \leq \frac{n}{m} + \frac{n}{m} \leq \frac{n}{l};$$

besides, depth$(ab) = \max\{$depth(a), depth$(b)\} \geq m \geq l$; then $ab \in V_l$, and the claim is proved.

Finally (12) follows from the fact that for every $a \in \Gamma(p)$ with $a \neq 0$, if $\text{depth}(a) = m$, then $a \notin V_{m+1}$. □

Kiltinen [7] and Arnautov [1] proved that every infinite field admits a nondiscrete, Hausdorff field topology. In [13, p. 84], a ring topology on an infinite algebraic field is given by means of Kiltinen's method. In this topology, a neighborhood basis at zero is defined recursively. We have not been able to find in the literature a more explicit example of a ring topology on $\Gamma(p)$.

We can easily get from $\mathscr{T}$ a field topology on $\Gamma(p)$ using as a basis of zero neighborhoods the family

$$\left\{\frac{V_m}{1 + V_m}\right\}_{m \geq 2}$$

(see [14, p. 18; 12, p. 32; 4; 5]). Let us describe some properties of the topology $\mathscr{T}$.

Lemma 6. *A sequence of elements $(b_n)_{n \in \mathbb{N}}$ of $\Gamma(p)$ is a Cauchy sequence with respect to the ring topology $\mathscr{T}$ if and only if* $\lim_{n \to \infty}(b_{n+1} - b_n) = 0$.

Proof. It is an immediate consequence of the fact that each $V_m \in \mathscr{B}$ is an additive subgroup of $\Gamma(p)$. □

We recall that a subset S of a commutative topological ring R is bounded if given any neighborhood V of zero, there exists a neighborhood U of zero such that $SU \subseteq V$. If R is a field, this is equivalent to saying that given any neighborhood V of zero, there exists a nonzero element $x \in R$ such that $Sx \subset V$ (see [10; 12, Theorem 3, p. 42; 14, Lemma 12, p. 26]). A ring topology on R is *locally bounded* if there is a bounded neighborhood of zero. The field $\Gamma(p)$ does not admit any proper locally bounded ring topology (see [7, Theorem 6.1]). Consequently, our topology $\mathscr{T}$ is locally unbounded. This fact can also be deduced from the lemma below.

Lemma 7. *For each neighborhood $V_m \in \mathscr{B}$ and each nonzero element $a \in \Gamma(p)$, $aV_m \nsubseteq V_{m+1}$. Therefore V_m is not a bounded set.*

Proof. Suppose there exists an a such that $aV_m \subseteq V_{m+1}$. Let n be an integer such that $n > \text{height}(a)$, $n > m$ and m divides n. Let $v \in G_n$ which satisfies $\text{wd}(v) = n/m$; since $\text{depth}(v) = n > m$, $v \in V_m$. But we have $av \in G_n$ and $\text{wd}(av) = n/m$, thus $av \notin V_{m+1}$, and hence $aV_m \nsubseteq V_{m+1}$, a contradiction. Consequently, such an a does not exist. □

We recall that a nonzero element a of a topological ring is topologically nilpotent if $\lim_{n \to \infty} a^n = 0$. In $(\Gamma(p), \mathscr{T})$ there is not any topologically nilpotent element, because for all $a \in \Gamma(p)$ the sequence $(a^n)_{n \in \mathbb{N}}$ is cyclic. Let $(\alpha_n)_{n \in \mathbb{N}}$ be the sequence of primitive elements defined in (2); we can easily check that $\lim_{n \to \infty} \alpha_n = 0$.

Lemma 8. *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonzero elements of $\Gamma(p)$ that converges to zero. Then* $\lim_{n\to\infty} \text{degree}(b_n) = +\infty$.

Proof. It suffices to consider that, for every $V_m \in \mathscr{B}$, the intersection $V_m \cap GF(p^{(m-1)!}) = \{0\}$. Then for all $a \in V_m$ with $a \neq 0$, we have degree$(a) > (m-1)!$. □

The above lemma shows the close relation between the algebraic structure of $\Gamma(p)$ and the ring topology $\mathscr{T}$. Its converse is not true at all. If (b_n) is a sequence of nonzero elements such that $\lim_{n\to\infty} b_n = 0$, we have that $\lim_{n\to\infty} \text{degree}(b_n^{-1}) = \lim_{n\to\infty} \text{degree}(b_n) = +\infty$, but $\lim b_n^{-1}$ does not exist.

3. The completion of $\Gamma(p)$

The field $\Gamma(p)$ is not complete with respect to the ring topology $\mathscr{T}$. For example, given the sequence $(\alpha_n)_{n\in\mathbb{N}}$ of primitive elements defined in (2), the sequence $(\beta_n)_{n\in\mathbb{N}}$ whose terms are defined as $\beta_n = \sum_{i=1}^{n} \alpha_i$ is a Cauchy sequence; but it is clear that (β_n) has no limit in $\Gamma(p)$. We shall consider its completion, which will be, as usual, the quotient ring of the ring of Cauchy sequences by the ideal of all sequences converging to zero. We say that two Cauchy sequences are equivalent if they represent the same element in this quotient ring. In order to describe this completion in an easier way, we will need the next lemma.

Since $\Gamma(p) = GF(p^{n!}) \otimes H_{n+1}$, for all $a \in \Gamma(p)$ there are $c \in GF(p^{n!})$ and $d \in H_{n+1}$ such that $a = c + d$, c and d being unique; we can define for each natural number $n \in \mathbb{N}$ the following map:

$$\begin{aligned} \pi_n : \Gamma(p) &\to GF(p^{n!}) \\ a &\to \pi_n(a) = c. \end{aligned} \tag{13}$$

It is clear that if $n \geq m$ then $\pi_m \circ \pi_n = \pi_m$.

Lemma 9. *Let (b_n) be a Cauchy sequence in $\Gamma(p)$. There exists a unique Cauchy sequence (a_n) which is equivalent to (b_n), satisfies $a_n \in GF(p^{n!})$ for all $n \in \mathbb{N}$ and $\pi_n(a_m) = a_n$ for all $m \geq n$.*

Proof. For every neighborhood V_{m+1} in $\mathscr{B}$, there is a natural number $N(m)$ such that $b_i - b_j \in V_{m+1}$ whenever $i, j \geq N(m)$. We may assume the sequence $N(m)$ to be strictly increasing with m; in particular $N(m) \geq m$. For all $m \in \mathbb{N}$, we define $a_m = \pi_m(b_{N(m)}) \in GF(p^{m!})$. Let us see that $\pi_m(a_n) = a_m$ for all $n \geq m$; it sufffices to show that $\pi_m(a_{m+1}) = a_m$. We have $b_{N(m+1)} - b_{N(m)} \in V_{m+1} \subset H_{m+1}$, and so $\pi_m(b_{N(m+1)}) = \pi_m(b_{N(m)})$, thus $\pi_m(a_{m+1}) = a_m$.

Let us see now that (a_n) is a Cauchy sequence equivalent to (b_n). Given $V_m \in \mathscr{B}$, for every natural number k such that $k > N(m)$ and

$$k > \text{height}(b_{N(m)}), \tag{14}$$

we have $a_k = \pi_k(b_{N(k)})$. We can split

$$b_k - a_k = (b_k - b_{N(m)}) + (b_{N(m)} - a_k). \tag{15}$$

Since $k > N(m)$ and $N(k) > N(m)$, we have $b_k - b_{N(m)} \in V_m$ and $b_{N(m)} - b_{N(k)} \in V_m$. Considering (14) and the structure of V_m in (7), we deduce $b_{N(m)} - \pi_k(b_{N(k)}) \in V_m$, that is, $b_{N(m)} - a_k \in V_m$. Then, (15) implies that $b_k - a_k \in V_m + V_m = V_m$. Consequently, (a_n) is a Cauchy sequence equivalent to (b_n).

Let us show the uniqueness. Let (a_n') be a different Cauchy sequence satisfying $\pi_n(a_{n+1}') = a_n'$ for all $n \in \mathbb{N}$. If $a_{n_0}' \neq a_{n_0}$, then $\pi_{n_0}(a_n') \neq \pi_{n_0}(a_n)$ for all $n \geq n_0$; hence $a_n' - a_n \notin V_{n_0+1}$, and so they are not equivalent. □

The situation described in the previous lemma is similar to that of the p-adic completion of the rational field $\mathbb{Q}$; see [8, Theorem 2, p. 11]. We shall denote by Λ_p the completion ring of $(\Gamma(p), \mathcal{T})$. Throughout the rest of the paper, $\sum_{n=1}^{\infty} a_n$ will mean the element in Λ_p which is the limit of the Cauchy sequence $(\sum_{n=1}^{m} a_n)_{m \in \mathbb{N}}$, in case it exists. Notice that whenever a sequence $(a_n)_{n \in \mathbb{N}}$ verifies that $a_n \in G_n$ and $\lim \mathrm{wd}(a_n)/n = 0$, the series $\sum_{n=1}^{\infty} a_n$ converges. Finally, let us describe the elements of Λ_p.

Theorem 10. *Each element $a \in \Lambda_p$ can be represented in a unique way as a series*

$$a = \sum_{n=1}^{\infty} a_n \quad \text{where } a_n \in G_n \text{ and } \lim_{n \to \infty} \frac{\mathrm{wd}(a_n)}{n} = 0.$$

Proof. Let $b \in \Lambda_p$. Applying lemma 9, we see that there is a Cauchy sequence (b_n) that satisfies $b_n \in GF(p^{n'})$, $\pi_n(b_{n+1}) = b_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} b_n = b$. For every integer $n \geq 2$, we set $a_n = b_n - b_{n-1} \in G_n$, and we let $a_1 = b_1 \in G_1$. Thus, $b_n = \sum_{m=1}^{n} a_m$. Since (b_n) is a Cauchy sequence, the series $\sum_{m=1}^{\infty} a_m$ converges and its sum is $\sum_{m=1}^{\infty} a_m = b$. Besides, $\lim a_m = 0$. Thus, for every $V_j \in \mathcal{B}$ there is $n_0 \in \mathbb{N}$ such that $a_n \in V_j$ for all $n \geq n_0$; hence, considering that $a_n \in G_n$, we have $\mathrm{wd}(a_n) \leq n/j$. Therefore $\mathrm{wd}(a_n)/n \leq 1/j$ for all $n \geq n_0$, and we have showed that $\lim \mathrm{wd}(a_n)/n = 0$. The uniqueness of this series follows from Lemma 9. □

We can extend the topology $\mathcal{T}$ from $\Gamma(p)$ to Λ_p; for convenience, we shall denote this extended topology also by $\mathcal{T}$. A fundamental system of neighborhoods of zero will be $\bar{\mathcal{B}} = \{\bar{V}_m\}_{m \in \mathbb{N}}$, where $\bar{V}_m$ is the closure of $V_m \in \mathcal{B}$ in our new topology on Λ_p (see [12, Theorem 5, p. 175]). It is easily checked that the neighborhoods in the basis $\bar{\mathcal{B}}$ will be the following sets:

$$\bar{V}_m = \left\{ a = \sum_{n=m}^{\infty} a_n \in \Lambda_p, \text{ where } a_n \in G_n \text{ and } \mathrm{wd}(a_n) \leq \frac{n}{m} \right\}. \tag{16}$$

We can also extend the "depth" function, in the obvious way to the whole ring Λ_p. Every nonzero element $a \in \bar{V}_m$ has depth greater than or equal to m. The ring

Λ_p endowed with the extended topology $\mathscr{T}$ is again a topological ring. Each neighborhood $\bar{V}_m$ from the basis $\tilde{\mathscr{B}}$ is an additive subgroup of $(\Lambda_p, +)$, and it is an open and closed set for the topology $\mathscr{T}$. Consequently, $(\Lambda_p, \mathscr{T})$ is a totally disconnected topological space. This completion has the following properties:

- A sequence (a_n) in Λ_p converges if and only if it is a Cauchy sequence, and if and only if $\lim(a_{n+1} - a_n) = 0$.
- A series $\sum_{n=1}^{\infty} b_n$ converges if and only if $\lim b_n = 0$.

Let us describe the structure of the ring Λ_p.

Proposition 11. *The ring Λ_p is an integral domain, in which the only invertible elements are those in $\Gamma(p)\setminus\{0\}$.*

Proof. Let $a, b \in \Lambda_p\setminus\{0\}$. Applying theorem 10, they can be expressed as

$$a = \sum_{n=1}^{\infty} a_n, \quad b = \sum_{n=1}^{\infty} b_n, \qquad \text{where } a_n, b_n \in G_n \text{ for all } n \in \mathbb{N}.$$

If $a \in \Gamma(p)$ or $b \in \Gamma(p)$, then clearly $ab \neq 0$. Thus, we assume that $a, b \in \Lambda_p\setminus\Gamma(p)$, and, therefore, there are infinitely many nonzero a_n and b_n. Since $\lim_{n\to\infty} \mathrm{wd}(a_n)/n = \lim_{n\to\infty} \mathrm{wd}(b_n)/n = 0$, there is $n_0 \in \mathbb{N}$ such that $\mathrm{wd}(a_n) < n/2$ and $\mathrm{wd}(b_n) < n/2$ for all $n \geq n_0$. We choose an integer j that satisfies $j > \max\{n_0, \mathrm{depth}(a), \mathrm{depth}(b)\}$ and $a_j \neq 0$. We now suppose that $b_j \neq 0$ (if $b_j = 0$ the proof follows similarly). For all $n < j$ we have $b_n a_j, b_j a_n \in G_j$. As $\mathrm{wd}(a_j) + \mathrm{wd}(b_j) < j$, then $a_j b_j \in G_j$. Therefore

$$c_j = \sum_{n=1}^{j-1} a_n b_j + \sum_{n=1}^{j-1} b_n a_j + a_j b_j \in G_j,$$

and since

$$\begin{aligned}\mathrm{wd}(a_j b_j) = \mathrm{wd}(a_j) + \mathrm{wd}(b_j) &> \max\{\mathrm{wd}(a_j), \mathrm{wd}(b_j)\} \\ &= \max\left\{\mathrm{wd}\left(\sum_{n=1}^{j-1} b_n a_j\right), \mathrm{wd}\left(\sum_{n=1}^{j-1} a_n b_j\right)\right\},\end{aligned}$$

we have $c_j \neq 0$. Since (see [2, III.6, Exercise 5])

$$\sum_{j=1}^{n} c_j = \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} b_j\right) \to ab,$$

it follows that $\sum_{j=1}^{\infty} c_j$ is the canonical representation of ab and that $ab \neq 0$; i.e., Λ_p is an integral domain. Since, also, $ab \neq 1$, the elements in $\Lambda_p\setminus\Gamma(p)$ are not units of the ring. □

The above proposition is related to problem 3 in [14, p. 251]. Λ_p is not locally bounded, since the intersection of a bounded neighborhood in Λ_p with $\Gamma(p)$ would be a bounded neighborhood in $\Gamma(p)$ [12, Theorem 4.1(7)].

4. A family of continuous functions

In every field with an absolute value $(K, |\,|)$, we can find a neighborhood of zero W such that for all $a \in W$, the sequence of powers $(a^m)_{m \in \mathbb{N}}$ converges to zero; for example, we may take $W = \{a \in K: |a| < 1/2\}$. This is the reason why it requires very few restrictive conditions over the "a_n" coefficients to guarantee the convergence of a power series $\sum_{n=0}^{\infty} a_n X^n$ is a neighborhood of zero. But in Λ_p we cannot find such a neighborhood W. Since $\Gamma(p)$ is dense in its completion, each neighborhood of zero in Λ_p contains elements that are not topologically nilpotent. Therefore, the theory of convergent power series in $(\Lambda_p, \mathscr{T})$ seems to be more restrictive than the case of a valuation field, and we shall not consider it. Nevertheless, we shall study a class of functions that can be used to construct convergent series.

Definition 3. The function $h: \Lambda_p \to \Lambda_p$ is defined as follows:
- If $b \in G_1$, then $h(b) = b\alpha_2$.
- For each $n \geq 2$, and every $a \in G_n$, let $Q_a(X) \in GF(p^{(n-1)!})[X]$ be the polynomial defined in (3) such that $a = Q_a(\alpha_n)$; we then define $h(a) = Q_a(\alpha_{n+1})$.
- For an arbitrary element $a \in \Lambda_p$ such that $a = \sum_{n=1}^{\infty} a_n$, where $a_n \in G_n$, we define $h(a) = \sum_{n=1}^{\infty} h(a_n)$.

Notice that for each n the inclusion $h(G_n) \subset G_{n+1}$ holds. Given any ring R containing the field $GF(p)$, we say that a map $g: R \to R$ is p-linear transformation if it is a linear transformation of R as vector space over $GF(p)$. The function h is clearly a p-linear transformation. Besides, the function h is continuous. It is not difficult to see that $h(\bar{V}_n) \subset \bar{V}_n$ for all $n \in \mathbb{N}$. Therefore, h is continuous at zero; then, since h is an additive group homomorphism, h is continuous at every $a \in \Lambda_p$.

We consider now the set of functions obtained by composing h with itself:

$$h^{(k)} = \overbrace{h \circ h \circ \cdots \circ h}^{k \text{ terms}}; \tag{17}$$

each of these functions is continuous and p-linear. Let us see that the set of functions $\{h^{(k)}\}_{k \in \mathbb{N}}$ has good convergence properties.

Proposition 12. *For every element $a \in \Lambda_p$, we have $\lim_{k \to \infty} h^{(k)}(a) = 0$.*

Proof. Let $a \in \Lambda_p$, with $a \neq 0$ and $\operatorname{depth}(a) = m$. The element a can be written as $a = \sum_{n=m}^{\infty} a_n$, where $a_n \in G_n$, $\operatorname{wd}(a_n) < n$ for all n and $\lim_{n \to \infty} \operatorname{wd}(a_n)/n = 0$. Then $h^{(k)}(a) = \sum_{n=m}^{\infty} h^{(k)}(a_n)$, with $h^{(k)}(a_n) \in G_{k+n}$. Given a neighborhood $\bar{V}_j \in \tilde{\mathscr{B}}$, there is a positive integer n_0 such that for all $n \geq n_0$ we have $\operatorname{wd}(a_n) \leq n/j$. Then, $\operatorname{wd}(h^{(k)}(a_n)) = \operatorname{wd}(a_n) \leq n/j \leq (k+n)/j$. Besides, for all $n \leq n_0$ and for every integer k that satisfies $k \geq jn_0$, we have $\operatorname{wd}(h^{(k)}(a_n)) = \operatorname{wd}(a_n) < n \leq n_0 \leq k/j < (k+n)/j$.

Finally, $\text{depth}(h^{(k)}(a)) = k + m$. Thus $h^{(k)}(a) \in \bar{V}_j$ for all k such that $k \geq jn_0$ and $k + m \geq j$, and so $\lim_{n\to\infty} h^{(k)}(a) = 0$. □

Therefore, the series $\sum_{n=1}^{\infty} h^{(n)}(x)$ converges at every value $x \in \Lambda_p$. Besides, we can get a large family of convergent series; for this purpose we need the following results.

Lemma 13. *For each $k \in \mathbb{N}$ the set*

$$L_k = \left\{ b = \sum_{n=1}^{\infty} b_n \in \Lambda_p, \text{ where } b_n \in G_n, \text{ and for all } n \in \mathbb{N}\ \text{wd}(b_n) < k \right\}$$

is bounded.

Proof. We fix $k \in \mathbb{N}$. Let $\bar{V}_m$ be a neighborhood of zero in $\tilde{\mathscr{B}}$. Let $\alpha_{km} \in G_{km}$ be the primitive element defined in (2) with the index km. Our aim is to prove that for all $b \in L_k$ the product $b\alpha_{km}$ belongs to $\bar{V}_m$. Since $b = \sum_{n=1}^{\infty} b_n$,

$$b\alpha_{km} = \left(\sum_{n=1}^{km} b_n \right) \alpha_{km} + \sum_{n=km+1}^{\infty} b_n \alpha_{km}.$$

We have $\sum_{n=1}^{km} b_n \alpha_{km} \in G_{km}$, and, for all $n > km$, the product $b_n \alpha_{km} \in G_n$. Hence, $\text{depth}(b\alpha_{km}) \geq km$. We now look at the weak-degree. Assuming that $b_{km} \neq 0$ (the case $b_{km} = 0$ is treated analogously),

$$\text{wd}\left(\left(\sum_{n=1}^{km} b_n \right) \alpha_{km} \right) = \text{wd}(b_{km}\alpha_{km}) = \text{wd}(b_{km}) + \text{wd}(\alpha_{km}) \leq (k-1) + 1 = \frac{km}{m},$$

and for all $n > km$ we have $\text{wd}(b_n \alpha_{km}) = \text{wd}(b_n) \leq k - 1 < n/m$. Thus, $b\alpha_{km} \in \bar{V}_m$ and $L_k \alpha_{km} \subset \bar{V}_m$ holds. Therefore, L_k is bounded. □

Proposition 14. *For each $x \in \Lambda_p$ we consider the series $g(x) := \sum_{n=1}^{\infty} b_n h^{(n)}(x)$. If there exists a set L_k such that $b_n \in L_k$ for all $n \in \mathbb{N}$, then the series converges for all $x \in \Lambda_p$. Moreover, the function $g: \Lambda_p \to \Lambda_p$ is a p-linear transformation continuous at all $x \in \Lambda_p$.*

Proof. By Proposition 12, every $x \in \Lambda_p$ satisfies $\lim_{n\to\infty} h^{(n)}(x) = 0$. The set $B = \{b_n\}_{n \in \mathbb{N}}$ is bounded because it is contained in L_k. Thus $\lim_{n\to\infty} b_n h^{(n)}(x) = 0$ and the series $\sum_{n=1}^{\infty} b_n h^{(n)}(x)$ converges. Since each function $h^{(n)}$ is p-linear, g is also p-linear. Therefore, in order to prove the continuity of g, it suffices to prove that g is continuous at zero. As B is a bounded set, given any neighborhood of zero $\bar{V}_m$ in $\tilde{\mathscr{B}}$, there exists another $\bar{V}_j \in \tilde{\mathscr{B}}$ such that $B\bar{V}_j \subseteq \bar{V}_m$. Since $h^{(n)}(\bar{V}_j) \subset \bar{V}_j$ for all $n \in \mathbb{N}$, then $b_n h^{(n)}(\bar{V}_j) \subset b_n \bar{V}_j \subseteq \bar{V}_m$ for all $n \in \mathbb{N}$. Thus, considering that $\bar{V}_m$ is a closed additive subgroup in a complete ring, for all $x \in \bar{V}_j$ we have $g(x) = \sum_{n=1}^{\infty} b_n h^{(n)}(x) \in \bar{V}_m$ and $g(\bar{V}_j) \subseteq \bar{V}_m$. Consequently, g is continuous at zero. □

5. A similar topology with nontrivial continuous automorphisms

In this section we shall give another example of a ring topology over $\Gamma(p)$, very similar to the above one $\mathscr{T}$, but strictly finer. We define a fundamental system of neighborhoods at zero. For all natural numbers m we set.

$$W_m = \left\{ \sum_{n \geq m} a_n \in \Gamma(p) \text{: for all } n,\ a_n \in G_n \text{ and } (\mathrm{wd}(a_n))^m \leq n \right\}. \tag{18}$$

This basis of neighborhoods will be denoted by $\hat{\mathscr{B}} = \{W_m\}_{m \in \mathbb{N}}$. Let us notice that, for all $n > 1$, the inequality in (18) can be written as

$$\frac{\log(\mathrm{wd}(a_n))}{\log(n)} \leq \frac{1}{m}. \tag{19}$$

Following an argument similar to that of Theorem 5, it can be shown that $\hat{\mathscr{B}}$ is a fundamental system of neighborhoods of zero for a Hausdorff ring topology on $\Gamma(p)$, that will be denoted by $\hat{\mathscr{T}}$. It is easy to see that for all integer $m \geq 2$ we have $W_m \subset V_m$, and V_m is not contained in W_2 for any m. Hence, the topology $\hat{\mathscr{T}}$ is strictly finer than $\mathscr{T}$. We can study the topology $\hat{\mathscr{T}}$ as we did with $\mathscr{T}$. In this new topology the completion is the following ring:

$$\Upsilon_p := \left\{ b = \sum_{n=1}^{\infty} b_n \text{: } b_n \in G_n \text{ and } \lim_{n \to \infty} \frac{\log(\mathrm{wd}(b_n))}{\log(n)} = 0 \right\}.$$

We have the strict inclusion $\Upsilon_p \subset \Lambda_p$. Moreover, in this ring Υ_p we shall study another family of continuous functions that we shall call 'suitable p-functions". We recall that a polynomial of the form

$$\sum_{i=0}^{n} a_i X^{p^i} \in GF(p^m)[X]$$

is called a p-polynomial, or a linearized polynomial over $GF(p^m)$. A function $f: GF(p^m) \to GF(p^m)$ is a p-linear transformation if and only if there exists a p-polynomial $H(X) \in GF(p^m)[X]$ such that $f(a) = H(a)$ for all $a \in GF(p^m)$. Moreover, the p-polynomial H can be chosen with degree less than p^m. Under this restriction of degree, for each f the corresponding H is unique (see [3, Theorem 1.15, p. 14]).

Definition 4. A function $F: \Gamma(p) \to \Gamma(p)$ is called a suitable p-function if it is a p-linear transformation and $F(GF(p^{n!})) \subseteq GF(p^{n!})$ for all $n \in \mathbb{N}$.

Let F be a suitable p-function. For all $n \in \mathbb{N}$, F can be restricted to the Galois field $GF(p^{n!})$ obtaining

$$F_n := F|_{GF(p^{n!})}: GF(p^{n!}) \to GF(p^{n!}),$$

where each F_n is a p-linear transformation. Consequently, for each n there is p-polynomial $H_n(X) \in GF(p^{n!})[X]$ such that $F(a) = F_n(a) = H_n(a)$ for all $a \in GF(p^{n!})$, and if $H_n(X) \neq 0$, $\deg(H_n(X)) = p^{r_n}$ satisfies $0 \leq r_n < n!$. Since every element $b \in \Gamma(p)$ can be written as a sum $b = \sum_{n=1}^{k} b_n$, where $b_n \in G_n$, $F(b)$ can be written as

$$F(b) = \sum_{n=1}^{k} F_n(b_n).$$

For every natural number n, we can get the polynomial H_n from H_{n+1} as the remainder of the euclidean division

$$H_{n+1} = Q_n(X)(X^{p^{n!}} - X) + H_n(X).$$

A suitable p-function F does not need to be a polynomial function; in fact, it is obvious that F is not a polynomial function if and only if $\lim_{n\to\infty} r_n = +\infty$. The following result informs us about the continuity of F.

Proposition 15. *Let F be the above function. If*

$$\lim_{n\to\infty} \frac{r_n}{\log(n)} = 0,$$

then F is continuous at all $x \in \Gamma(p)$. Furthermore, F can be extended, in the natural way, to a function $\bar{F}: \Upsilon_p \to \Upsilon_p$ which is p-linear and continuous at all $x \in \Upsilon_p$.

Proof. It suffices to show that F is continuous at zero, because it is a p-linear function. Let W_k be a neighborhood from the basis $\hat{\mathscr{B}}$ defined in (18). By hypothesis we have

$$0 = \lim_{n\to\infty} \frac{r_n}{\log n} \log(p) = \lim_{n\to\infty} \frac{\log(p^{r_n})}{\log n} = \lim_{n\to\infty} \frac{\log(\deg(H_n))}{\log n}.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\log(\deg(H_n))}{\log(n)} < \frac{1}{2k}$$

for all $n \geq n_0$, that is, $(\deg(H_n))^{2k} < n$. Let $j = \max\{2k, n_0\}$, and the W_j be the corresponding neighborhood in $\hat{\mathscr{B}}$. Every $b \in W_j$ can be written as $\sum_{n=j}^{l} b_n$, where $b_n \in G_n$, and for all $n \geq j$, if $b_n \neq 0$ then $(\mathrm{wd}(b_n))^j \leq n$. Thus, if $b_n \neq 0$ we have $(\mathrm{wd}(F_n(b_n)))^k = (\deg(H_n)\mathrm{wd}(b_n))^k = (\deg(H_n))^k(\mathrm{wd}(b_n))^k < n^{1/2}n^{1/2} = n$. We have proved also that $F_n(b_n) \in G_n$; consequently $F(b) = \sum_{n=j}^{l} F_n(b_n) \in W_k$ and $F(W_j) \subset W_k$. Hence, F is continuous at zero.

The function F can be easily extended to a function $\bar{F}$ over the ring Υ_p as follows. If $b = \sum_{n=1}^{\infty} b_n \in \Upsilon_p$, where $b_n \in G_n$, then $\bar{F}(b) = \sum_{n=1}^{\infty} F(b_n)$. $\bar{F}$ is clearly a continuous p-linear function (cf. Proposition 14). □

A particular case of especial importance is given when F is an automorphism of the field $\Gamma(p)$; then, every function F_n is also an automorphism of the finite field $GF(p^{n!})$ (see [3, Theorem 1.11, p. 10] for a description of the automorphisms of a finite field). For all natural numbers n there exists $r_n \in \mathbb{N}$, $0 \leq r_n < n!$ such that for every

$a \in GF(p^{n!})$ we have

$$F_n(a) = a^{p^{r_n}}.$$

That is, F_n is a polynomial function represented by the polynomial

$$H_n(X) = X^{p^{r_n}}.$$

We can check easily that each r_n is the remainder of the division of r_{n+1} by $n!$, i.e., $r_{n+1} = q_n n! + r_n$. Hence, we have the following consequences of the above proposition.

Corollary 16. *Let $F : \Gamma(p) \to \Gamma(p)$ be the above automorphism. If*

$$\lim_{n\to\infty} \frac{r_n}{\log(n)} = 0,$$

then F is continuous at every $a \in \Gamma(p)$.

6. A more general class of fields

Topologies similar to both described above can be constructed in every infinite algebraic extension of a finite field (these fields are described in [3]), and in a more general class of fields. Let $(s_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of natural numbers that satisfies $\lim s_n = +\infty$. Let $\{K_n\}_{n\in\mathbb{N}}$ be a family of fields such that, for each $n \geq 2$,

- $K_{n-1} \subset K_n$;
- there exists $\sigma_n \in K_n$ such that $K_n = K_{n-1}(\sigma_n)$;
- every field extension K_n/K_{n-1} has degree $[K_n : K_{n-1}] \geq s_n$.

We consider the field $K = \bigcup_{n\in\mathbb{N}} K_n$. We can define the functions depth, height and wd( ) in this field K as we did in Section 2, and we get again a ring topology with similar features. In this topological ring it also holds that a sequence of primitive elements converges to zero.

The field K constructed in this way and the field $\Gamma(p)$ are particular cases of algebraically unbounded rings, which are defined in [7, Section 4]. Kiltinen constructs his inductive ring topologies on the algebraically unbounded rings. The topologies we have introduced in this article are particular cases of weak inductive ring topologies [7, Corollary 8.2]. Let us see two examples.

Example. For every natural number let $\zeta_{n!} = e^{2\pi i/n!}$ be a primitive $n!$th root of unity, such that $(\zeta_{n!})^n = \zeta_{(n-1)!}$ for all n. We consider the field

$$K = \bigcup_{n\in\mathbb{N}} \mathbb{Q}(\zeta_{n!}).$$

We recall that, for all $n \geq 2$, the degree of the field extension $\mathbb{Q}(\zeta_{n!})/\mathbb{Q}(\zeta_{(n-1)!})$ is

$$[\mathbb{Q}(\zeta_{n!}):\mathbb{Q}(\zeta_{(n-1)!})] = \frac{\phi(n!)}{\phi((n-1)!)} = \begin{cases} n-1 & \text{if } n \text{ is prime,} \\ n & \text{if } n \text{ is not prime.} \end{cases}$$

Thus we can define on K a ring topology in which we have $\lim_{n\to\infty} \zeta_{n!} = 0$.

Example. For all integers $n \geq 2$ let $\beta_n = \sqrt[n!]{2}$ be a positive real root of the polynomial $X^{n!} - 2$. Let us consider the field

$$K_2 = \bigcup_{n \in \mathbb{N}} \mathbb{Q}(\beta_n).$$

The degree of each extension $\mathbb{Q}(\beta_n)/\mathbb{Q}(\beta_{n-1})$ is n. One can define a ring topology on K_2 in which we have $\lim_{n\to\infty} \sqrt[n!]{2} = 0$.

Acknowledgements

The author would like to thank Capi Corrales, who suggested many linguistic corrections to a preliminary draft of this paper. In addition, I thank Niel Shell and the referee for several suggestions and corrections.

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